Lecture #24

Last time: "Simple Interest"

Initial data:

- Lender (L) lends $P$ to Borrower (B)
- interest rate (\%) \leftarrow use APR when it is a rate per year
- term (t) - length of time

At the end the Borrower (B) has to pay $F$ back to Lender (L)

Simple Interest Formula:

\[ F = P + P \cdot r \cdot t = P \cdot (1 + t \cdot \% ) \]

Another way to write it is via \[ I = F - P \]

\[ I = t \cdot \% \cdot P \]

Ex1: Alice bought a 5-year government bond at APR = 4% with a face value (= amount she will get back after 5 years) of $6,000.

How much did she pay originally?

\[ F = P \cdot (1 + t \cdot \%) \Rightarrow P = \frac{F}{1 + t \cdot \%} \]

\[ F = 6,000, \quad r = \frac{4}{100}, \quad t = 5 \]

Answer: $5,000 is the amount Alice spent on this bond originally.

Ex2: John took a loan from a bank at the amount of $4,000 for 5 years with simple interest. At the end he paid back to the bank $5,000. Find the APR.

\[ F = P \cdot (1 + t \cdot \%) \Rightarrow 1 + t \cdot \% = \frac{F}{P} \]

\[ F = 5,000, \quad P = 4,000, \quad t = 5 \]

As \[ \frac{t}{5} = 0.05 \Rightarrow APR was 5\% \]
Today: "Compound Interest"  (§ 10.3)

Basic Idea: Previously accumulated interest generates an interest

**Compound Interest Formula:** \[ F = P \cdot (1+r)^t \]

Let us consider the same example as at the end of last lecture, but in the setting of compound interest.

**Ex3:** Bob took a loan of $6000 at compound annual interest at APR = 5% and for the length of 4 years. What will he pay at the end?

- First, at the moment he borrows money, he owes $6000
- After 1st year, he owes $6,000 + \( \underbrace{0.05 \cdot 6000} \) = $6,300
- After 2nd year, he owes $6,300 + \( \underbrace{0.05 \cdot 6,300} \) = $6,615
- After 3rd year, he owes $6,615 + \( \underbrace{0.05 \cdot 6,615} \) = $6,945.75
- Finally after 4th year, he owes $6,945.75 + \( \underbrace{0.05 \cdot 6,945.75} \) \( \approx \) $7,293.04

Bob has to pay back $7,293.04, while as we saw last time at simple interest he would have to pay back only $7,200.

At this example, the difference is not really big.

But if we change the length of time to e.g. 20 years, then at simple interest he would have to pay back $12,000, while at compound interest - $15,919.75.
Ex 4: John invested $2000 into a CD (Certificate of Deposit) with an APR of 3.6% compounded annually. Find the future value (= cash value of CD at the end of its term) corresponding to \( t = 1, 5, 10, 15, 20 \).

- \( t = 1: \$2000 \cdot (1 + \frac{3.6}{100}) = \$2072 \)
- \( t = 5: \$2000 \cdot (1 + \frac{3.6}{100})^5 \approx \$2386 \)
- \( t = 10: \$2000 \cdot (1 + \frac{3.6}{100})^{10} \approx \$2848 \)
- \( t = 15: \$2000 \cdot (1 + \frac{3.6}{100})^{15} \approx \$3339 \)
- \( t = 20: \$2000 \cdot (1 + \frac{3.6}{100})^{20} \approx \$4057 \)

**Rule of 72**: It takes approximately \( \frac{72}{APR} \) to double your original investment.

In the above example, \( \frac{72}{3.6} = 20 \) and we see that future value after 20 years is almost twice original value.

**General Compounding**

There is no reason to assume that interest is compounded only once per year. In the example of CD's account, there are accounts with interest compounded semi-annually, quarterly, or monthly.

- **General compounding formula**: \( F = P \cdot (1 + \frac{r}{n})^{nt} \) - same as before

If it is e.g. a quarterly compound interest with APR = 3.6%, then it means the interest per quarter is \( \frac{1}{4} \cdot 3.6\% = 0.9\% \).
**Ex 5:** Consider the setting of Ex 4, but with interest compounded on the monthly basis. Find the future values after 1, 5, 10, 15, 20 years.

Since APR = 3.6% and one year consists of 12 months, it means that monthly interest is \( \frac{3.6\%}{12} = 0.3\% \).

- \( t = 1 \text{ year} = 12 \text{ monthly} \implies 2000 \cdot (1 + \frac{0.3}{100})^{12} \approx 2073.20 \)
- \( t = 5 \text{ years} = 60 \text{ monthly} \implies 2000 \cdot (1 + \frac{0.3}{100})^{60} \approx 2393.73 \)
- \( t = 10 \text{ years} = 120 \text{ monthly} \implies 2000 \cdot (1 + \frac{0.3}{100})^{120} \approx 2865.74 \)
- \( t = 15 \text{ years} = 180 \text{ monthly} \implies 2000 \cdot (1 + \frac{0.3}{100})^{180} \approx 3429.24 \)
- \( t = 20 \text{ years} = 240 \text{ monthly} \implies 2000 \cdot (1 + \frac{0.3}{100})^{240} \approx 4104.14 \)

**Remark:** After all, the difference is not that big!

- **Continuous Compounding**

  We started originally from the case, when interest was compounded once a year, but then also discussed that often it is compounded once per a shorter period of time.

  In the "limit", we got a situation when interest is compounded continuously.

  \[ F = P \cdot e^{rt} \]

  Here \( e \approx 2.718281828 \ldots \) - Euler's Number