Lecture #30

**Last time: Fibonacci numbers**

**Ex 1 (Exercise 13.1.8):** Given that $F_{32} = 2,178,303$ and $F_{33} = 3,524,578$,

(a) find $F_{34}$

(b) find $F_{35}$.

(a) $F_{34} = F_{33} + F_{32} = \frac{3524578}{2178303} = 5,702,887$

(b) $F_{35} = F_{34} + F_{33} = \frac{5702887}{3524578} = 9,227,465$

**Ex 2 (Exercise 13.1.13(c)):** Find the value of $\phi$ such that

$$F_1 + F_3 + F_5 + F_7 + \ldots + F_N = \phi$$

All numbers

When $N=3$,

$$F_1 + F_3 = 1 + 2 = 3 = F_4$$

When $N=5$,

$$F_1 + F_3 + F_5 = F_4 + F_5 = F_6$$

When $N=7$,

$$F_1 + F_3 + F_5 + F_7 = F_6 + F_7 = F_8$$

General assumption: $F_1 + F_3 + \ldots + F_{2n-1} + F_{2n} = F_{2n+1}$

So: $\phi = N + 1$.

**The "Golden Ratio"**

**Def.:** The golden ratio $\phi$ is defined as

$$\phi = \frac{1 + \sqrt{5}}{2}$$

Greek letter "phi"

**Prop.:** $\phi$ is an irrational number, with decimal approximation

$$\phi \approx 1.6180339887$$

**Golden property:** $\phi^2 = \phi + 1$.

As we saw last time, $\phi$ is a unique positive solution of (4). Explain again!
Lecture #30

- The Divine Proportion

The golden ratio was first considered by ancient Greeks. They figured out that an ideal proportional split of anything is given by golden ratio.

\[ \frac{B+S}{S} = \phi \]

If we denote \( \frac{B}{S} \) by \( x \), then we get \( x = 1 + \frac{1}{x} \) \( \Rightarrow \) \( x^2 = x + 1 \).

So \( x \) satisfies the golden property and \( x > 0 \) \( \Rightarrow \) \( x = \phi \).

Starting from ancient Greeks, the golden ratio has been used by painters, architects, designers, etc.

Examples: The ratio of sides of a credit card as well as the ratio of sides of a MacBook is \( \approx 1.6 \) which is almost \( \phi \).

the difference is really invisible to human eyes.

- Relation between Fibonacci numbers and the Golden Ratio

(i) Binet's Formula from last time can be written as:

\[ F_n = \left( \phi ^n - (1-\phi)^n \right) / \sqrt{5} \]

or \[ F_n = \left[ \phi^n / \sqrt{5} \right] \]

(ii) \( \phi^2 = \phi + 1 \) \( \Rightarrow \) \( \phi^3 = \phi^2 + \phi = 2\phi + 1 \) \( \Rightarrow \) \( \phi^4 = 2\phi^2 + \phi = 3\phi + 2 \) \( \Rightarrow \) \( \phi^5 = 3\phi^3 + 2\phi = 5\phi + 3 \)....

We see that \[ \phi^n = F_n\phi + F_{n-1} \]

(Just note that \( (F_n\phi + F_{n-1}) : \phi = F_n \), \( \phi^2 + F_n \phi = F_n (\phi + 1) + F_{n-1} \phi = (F_n + F_{n-1}) \phi + F_n \))
(iii) It turns out that the ratio of two consecutive Fibonacci numbers \( \frac{F_{n+1}}{F_n} \) "approaches" the golden ratio \( \phi \) as \( N \) becomes bigger and bigger.

N.B.: Mathematically, we say that \( \phi \) is a limit of \( \frac{F_{n+1}}{F_n} \) and write \( \frac{F_{n+1}}{F_n} \to \phi \).

**Ex 3 (Exercise 13.2.33)** Consider the quadratic equation

\[ F_n \cdot x^2 = F_{n-1} \cdot x + F_{n-2} \]

(a) Show that \( x = 1 \) is one of the two solutions of the equation.

(b) Without using the quadratic formula, find the second solution.

(a) For this part, we don't need to use standard formula

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

for the solutions of \( ax^2 + bx + c = 0 \).

Instead, let us just plug in \( x = 1 \):

\[ F_n \cdot x^2 |_{x=1} = F_n, \quad (F_{n-1} \cdot x + F_{n-2}) |_{x=1} = F_{n-1} + F_{n-2} = F_n \Rightarrow \text{Indeed a solution as } F_n = F_n \]

(b) By aforementioned formula \( x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) \( \Rightarrow x_1 + x_2 = -\frac{b}{a} \)

In our case:

\[ F_n \cdot x^2 = F_{n-1} \cdot x + F_{n-2} \Rightarrow F_n \cdot x^2 - F_{n-1} \cdot x - F_{n-2} = 0 \]

So: \( a = F_n, b = -F_{n-1}, c = -F_{n-2} \Rightarrow -\frac{b}{a} = \frac{F_{n-1}}{F_n} \).

Hence, \( x_1 + x_2 = \frac{F_{n-1}}{F_n} \). But according to (a) we have \( x_1 = 1 \) \( \Rightarrow x_2 = \frac{F_{n-1}}{F_n} - 1 = \frac{F_{n-1} - F_n}{F_n} = \frac{F_{n-2}}{F_n} \).